COMMENTS ON "A COMBINATORIAL VERSION OF THE GROTHENDIECK CONJECTURE"

Shinichi Mochizuki

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(1.) In the final sentence of Definition 1.1, (ii), the phrase "rank ≥ 2" should read "rank > 2".

(2.) In Example 2.5, the set of primes Σ should be assumed to be *nonempty*.

(3.) In the *second sentence* of Proposition 1.2, the phrase "*a vertex v_i (respectively, an edge e_i) of $\Pi_{\mathcal{G}}$*" should read "*a vertex v_i (respectively, an edge e_i) of $\mathbb{G}$*".

(4.) The arguments applied in Definition 1.4, (v), (vi), and Remarks 1.4.2, 1.4.3, and 1.4.4 to prove Theorem 1.6 are formulated in a somewhat confusing way and should be modified as follows:

(i) First of all, we remark that throughout the paper, as well as in the following discussion, a *"Galois"* finite étale covering is to be understood as being *connected*.

(ii) In the *second sentence* of Definition 1.4, (v), the *cuspidal* and *nodal* cases of the notion of a *purely totally ramified* covering are in fact *unnecessary* and *may be deleted*. Also, the terminology introduced in Definition 1.4, (vi), concerning finite étale coverings that *descend* is *unnecessary* and *may be deleted*.

(iii) The text of Remark 1.4.2 should be replaced by the following text:

Let $\mathcal{G}' \to \mathcal{G}$ be a Galois finite étale covering of degree a *positive power of* l, where $\mathcal{G}$ is of pro-Σ PSC-type, $\Sigma = \{l\}$. Then one verifies immediately that, if we assume further that the covering $\mathcal{G}' \to \mathcal{G}$ is *cyclic*, then $\mathcal{G}' \to \mathcal{G}$ is *cuspidally totally ramified* if and only if the inequality

$$\underline{r}(\mathcal{G}'') < l \cdot \underline{r}(\mathcal{G})$$

— where we write $\mathcal{G}' \to \mathcal{G}'' \to \mathcal{G}$ for the *unique* [up to isomorphism] *factorization* of the finite étale covering $\mathcal{G}' \to \mathcal{G}$ as a composite of finite étale coverings such that $\mathcal{G}'' \to \mathcal{G}$ is of degree l — is satisfied. Suppose further that $\mathcal{G}' \to \mathcal{G}$ is a [*not necessarily cyclic!*] $\Pi_{\mathcal{G}}^{\mathrm{unr}}$-*covering* [so $\underline{n}(\mathcal{G}') =$

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$\deg(\mathcal{G}'/\mathcal{G}) \cdot \underline{n}(\mathcal{G})]$. Then one verifies immediately that $\mathcal{G}' \to \mathcal{G}$ is *verticially purely totally ramified* if and only if the equality

$$\underline{i}(\mathcal{G}') = \deg(\mathcal{G}'/\mathcal{G}) \cdot (\underline{i}(\mathcal{G}) - 1) + 1$$

is satisfied. Also, we observe that this last inequality is equivalent to the following equality involving the expression "$\underline{i}(\dots) - \underline{n}(\dots)$" [cf. Remark 1.1.3]:

$$\underline{i}(\mathcal{G}') - \underline{n}(\mathcal{G}') = \deg(\mathcal{G}'/\mathcal{G}) \cdot (\underline{i}(\mathcal{G}) - \underline{n}(\mathcal{G}) - 1) + 1$$

(iv) The text of Remark 1.4.3 should be replaced by the following text:

Suppose that $\mathcal{G}$ is of pro-Σ PSC-type, $\Sigma = \{l\}$. Then one verifies immediately that the *cuspidal edge-like subgroups* of $\Pi_{\mathcal{G}}$ may be *characterized* as the *maximal* [cf. Proposition 1.2, (i)] closed subgroups $A \subseteq \Pi_{\mathcal{G}}$ isomorphic to $\mathbb{Z}_l$ which satisfy the following *condition*:

> for every characteristic open subgroup $\Pi_{\mathcal{G}'} \subseteq \Pi_{\mathcal{G}}$, if we write $\mathcal{G}' \to \mathcal{G}'' \to \mathcal{G}$ for the finite étale coverings corresponding to $\Pi_{\mathcal{G}'} \subseteq \Pi_{\mathcal{G}''} \overset{\text{def}}{=} A \cdot \Pi_{\mathcal{G}'} \subseteq \Pi_{\mathcal{G}}$, then the *cyclic* finite étale covering $\mathcal{G}' \to \mathcal{G}''$ is *cuspidally totally ramified*.

[Indeed, the *necessity* of this *characterization* is immediate from the definitions; the *sufficiency* of this *characterization* follows by observing that since the set of cusps of a finite étale covering of $\mathcal{G}$ is always *finite*, the above *condition* implies that there exists a *compatible system of cusps* of the various $\mathcal{G}'$ that arise, each of which is *stabilized* by the action of A.] On the other hand, in order to characterize the *unramified verticial subgroups* of $\Pi_{\mathcal{G}}^{\mathrm{unr}}$, it suffices — by considering *stabilizers* of *vertices* of underlying semi-graphs of finite étale $\Pi_{\mathcal{G}}^{\mathrm{unr}}$-coverings of $\mathcal{G}$ — to give a *functorial characterization* of the *set of vertices* of $\mathcal{G}$ [i.e., which may also be applied to finite étale $\Pi_{\mathcal{G}}^{\mathrm{unr}}$-coverings of $\mathcal{G}$]. This may be done, for *sturdy* $\mathcal{G}$, as follows. Write $M_{\mathcal{G}}^{\mathrm{unr}}$ for the *abelianization* of $\Pi_{\mathcal{G}}^{\mathrm{unr}}$. For each vertex v of the underlying semi-graph $\mathbb{G}$ of $\mathcal{G}$, write $M_{\mathcal{G}}^{\mathrm{unr}}[v] \subseteq M_{\mathcal{G}}^{\mathrm{unr}}$ for the image of the $\Pi_{\mathcal{G}}^{\mathrm{unr}}$-conjugacy class of unramified verticial subgroups of $\Pi_{\mathcal{G}}^{\mathrm{unr}}$ associated to v. Then one verifies immediately, by *constructing suitable abelian $\Pi_{\mathcal{G}}^{\mathrm{unr}}$-coverings* of $\mathcal{G}$ via suitable *gluing* operations [i.e., as in the proof of Proposition 1.2], that the inclusions $M_{\mathcal{G}}^{\mathrm{unr}}[v] \subseteq M_{\mathcal{G}}^{\mathrm{unr}}$ determine a *split injection*

$$\bigoplus_v M_{\mathcal{G}}^{\mathrm{unr}}[v] \hookrightarrow M_{\mathcal{G}}^{\mathrm{unr}}$$

[where v ranges over the vertices of $\mathbb{G}$], whose image we denote by $M_{\mathcal{G}}^{\mathrm{unr\text{-}vert}} \subseteq M_{\mathcal{G}}^{\mathrm{unr}}$. Now we consider *elementary abelian quotients*

$$\phi : M_{\mathcal{G}}^{\mathrm{unr}} \twoheadrightarrow Q$$

— i.e., where Q is an *elementary abelian group*. We *identify* such quotients whenever their *kernels coincide* and *order* such quotients by means of the

relation of *"domination"* [i.e., inclusion of kernels]. Then one verifies immediately that such a quotient $\phi : M_{\mathcal{G}}^{\mathrm{unr}} \twoheadrightarrow Q$ corresponds to a *vertically purely totally ramified* covering of $\mathcal{G}$ if and only if there exists a vertex v of $\mathbb{G}$ such that $\phi(M_{\mathcal{G}}^{\mathrm{unr}}[v]) = Q$, $\phi(M_{\mathcal{G}}^{\mathrm{unr}}[v']) = 0$ for all vertices $v' \neq v$ of $\mathbb{G}$. In particular, one concludes immediately that

> the elementary abelian quotients $\phi : M_{\mathcal{G}}^{\mathrm{unr}} \twoheadrightarrow Q$ whose restriction to $M_{\mathcal{G}}^{\mathrm{unr\text{-}vert}}$ surjects onto Q and has the same kernel as the quotient
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> $$M_{\mathcal{G}}^{\mathrm{unr\text{-}vert}} \;\twoheadrightarrow\; M_{\mathcal{G}}^{\mathrm{unr}}[v] \;\twoheadrightarrow\; M_{\mathcal{G}}^{\mathrm{unr}}[v] \otimes \mathbb{F}_l$$
>
> — where the first "$\twoheadrightarrow$" is the natural projection; the second "$\twoheadrightarrow$" is given by reduction modulo l — may be characterized as the *maximal quotients* [i.e., relative to the relation of domination] among those elementary abelian quotients of $M_{\mathcal{G}}^{\mathrm{unr}}$ that correspond to *vertically purely totally ramified* coverings of $\mathcal{G}$.

Thus, since $\mathcal{G}$ is *sturdy*, the *set of vertices* of $\mathcal{G}$ may be characterized as the *set of [nontrivial!] quotients* $M_{\mathcal{G}}^{\mathrm{unr\text{-}vert}} \twoheadrightarrow M_{\mathcal{G}}^{\mathrm{unr}}[v] \otimes \mathbb{F}_l$.

(v) The text of Remark 1.4.4 should be replaced by the following text:

Suppose that $\mathcal{G}$ is of pro-Σ PSC-type, where $\Sigma = \{l\}$, and that $\mathcal{G}$ is *noncuspidal*. Then, in the spirit of the *cuspidal* portion of Remark 1.4.3, we observe the following: One verifies immediately that the *nodal edge-like subgroups* of $\Pi_{\mathcal{G}}$ may be *characterized* as the *maximal* [cf. Proposition 1.2, (i)] closed subgroups $A \subseteq \Pi_{\mathcal{G}}$ isomorphic to $\mathbb{Z}_l$ which satisfy the following *condition*:

> for every characteristic open subgroup $\Pi_{\mathcal{G}'} \subseteq \Pi_{\mathcal{G}}$, if we write $\mathcal{G}' \to \mathcal{G}'' \to \mathcal{G}$ for the finite étale coverings corresponding to $\Pi_{\mathcal{G}'} \subseteq \Pi_{\mathcal{G}''} \stackrel{\mathrm{def}}{=} A \cdot \Pi_{\mathcal{G}'} \subseteq \Pi_{\mathcal{G}}$, then the *cyclic* finite étale covering $\mathcal{G}' \to \mathcal{G}''$ is *nodally totally ramified*.

Here, we note further that [one verifies immediately that] the finite étale covering $\mathcal{G}' \to \mathcal{G}''$ is *nodally totally ramified* if and only if it is *module-wise nodal*.

(vi) The text of the *second paragraph* of the proof of Theorem 1.6 should be replaced by the following text [which may be thought as being appended to the end of the *first paragraph* of the proof of Theorem 1.6]:

> Then the fact that α is *group-theoretically cuspidal* follows formally from the characterization of *cuspidal edge-like subgroups* given in Remark 1.4.3 and the characterization of *cuspidally totally ramified* cyclic finite étale coverings given in Remark 1.4.2.

(vii) The text of the *final paragraph* of the proof of Theorem 1.6 should be replaced by the following text [which may be thought of as a sort of *"easy version"*

of the argument given in the proof of the implication "(iii) $\Longrightarrow$ (i)" of [CbTpII], Proposition 1.5]:

> Finally, we consider assertion (iii). *Sufficiency* is immediate. On the other hand, *necessity* follows formally from the characterization of *unramified vertical subgroups* given in Remark 1.4.3 and the characterization of *verticially purely totally ramified* cyclic finite étale coverings given in Remark 1.4.2.

(5.) In Remarks 2.8.1, 2.8.2, one works "in the situation of Corollary 2.8", despite the fact that the assumption "$p \notin \Sigma$" in the statement of Corollary 2.8 is *not satisfied* in the situations considered in Remarks 2.8.1, 2.8.2. At first glance, this state of affairs may strike the reader as self-contradictory. The point, however, is that one thinks of Corollary 2.8 as being applied to the various *maximal pro-l quotients* of open subgroups of the geometric fundamental groups that appear in Remark 2.8.1, 2.8.2, i.e., that one takes the "Σ" of Corollary 2.8 to be $\{l\}$ for a suitable prime number l such that the assumptions in the statement of Corollary 2.8 are indeed satisfied.

(6.) In the first sentence of the fourth paragraph of the discussion entitled "Curves" in §0, the notation "$\overline{\mathcal{D}}_{g,r} \subseteq \overline{\mathcal{M}}_{g,r}$" should read "$\overline{\mathcal{D}}_{g,r} \subseteq \overline{\mathcal{C}}_{g,r}$".

Bibliography

[CbTpII] Y. Hoshi, S. Mochizuki, *Topics Surrounding the Combinatorial Anabelian Geometry of Hyperbolic Curves II: Tripods and Combinatorial Cuspidalization*, RIMS Preprint **1762** (November 2012).